The algebraic matroid of a Hadamard product of linear spaces and a new proof of Laman's Theorem

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Motivation

Definition

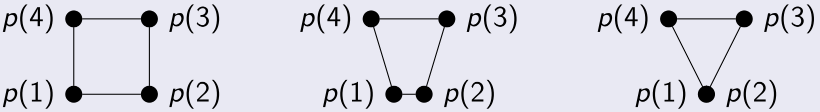
A *bar and joint framework in d dimensions* consists of

- a graph G, and
- a function $p : V(G) \to \mathbb{R}^d$.

Such frameworks can be *rigid* or *flexible*.

Example

Let G be the graph on vertex set $V = \{1, 2, 3, 4\}$ with edges $\{12, 23, 34, 14\}$. Below we give three functions $p : V \to \mathbb{R}^2$. The first two frameworks are flexible and the third one is rigid.



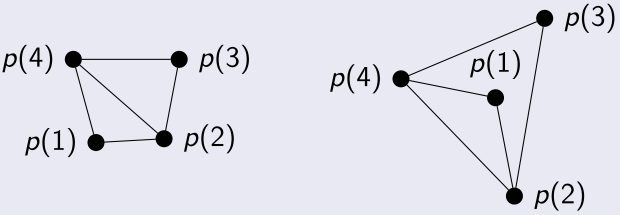
Generic rigidity

Definition

A graph G is *generically rigid in* $\mathbb{R}^d$ if for every generic $p : V \to \mathbb{R}^d$, the resulting framework is rigid. Such a graph is *minimal* if removing any edge destroys this property.

Example

Let G be the graph on vertex set $\{1, 2, 3, 4\}$ with all edges, aside from $\{1, 3\}$. For generic $p : \{1, 2, 3, 4\} \to \mathbb{R}^2$, the resulting framework is rigid.



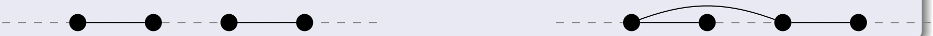
Motivating question

Question

Which graphs are (minimally) generically rigid in $\mathbb{R}^d$?

Proposition (Folklore)

A graph is generically rigid in $\mathbb{R}^1$ if and only if it is connected.



Theorem (Pollaczek-Geiringer 1927, "Laman's Theorem")

A graph G on n vertices is minimally generically rigid in $\mathbb{R}^2$ if and only if

1. *G has $2n - 3$ edges, and*
2. *every subgraph H of G on n' vertices has at most $2n' - 3$ edges.*

Such graphs are called Laman graphs.

Our main result will have Laman's theorem as an easy consequence.

Algebraic matroids

Each subset $S \subseteq E$ defines a coordinate projection $\pi_S : \mathbb{C}^E \to \mathbb{C}^S$.

Definition

Let $V \subseteq \mathbb{C}^E$ be an irreducible variety. A given $S \subseteq E$ is

1. *independent* if $\dim(\pi_S(V)) = |S|$,
2. *spanning* if $\dim(\pi_S(V)) = \dim(V)$, and
3. *a basis* if S is both independent and spanning.

The common combinatorial structure described by any one of these set systems is called the *algebraic matroid underlying* V.

Let $E = \{1,2,3\} \times \{1,2,3\}$ and $V \subseteq \mathbb{C}^E$ be the variety of 3×3 matrices of rank ≤ 1. Then $S := \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,3)\}$ is spanning, but not independent.

$$\pi_S(V) = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & \cdot \\ \cdot & \cdot & x_{33} \end{pmatrix} : x_{11}x_{22} - x_{21}x_{22} = 0 \right\}$$

Algebraic matroids in rigidity theory

Definition

Given a pair of integers $d \leq n$, the *Cayley-Menger variety of n points in $\mathbb{R}^d$*, denoted CM_n^d, is the affine variety embedded in $\mathbb{C}^{\binom{[n]}{2}}$ as the Zariski closure of the set of possible squared pairwise euclidean distances between n points in $\mathbb{R}^d$.

Example

Let $d = 2$. Then the ij coordinate of CM_n^2 is parameterized as $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$.

Observation

A graph $G = ([n], E)$ is generically rigid in $\mathbb{R}^d$ if and only if E is spanning in CM_n^d. Moreover, G is minimally generically rigid if and only if E is a basis of CM_n^d.

Algebraic matroids elsewhere

- Algebraic matroid underlying determinantal varieties useful in matrix completion (Király, Theran, Tomioka 2015)
- Combinatorial characterization of algebraic matroid underlying rank-two determinantal variety (B. 2017)
- Sufficient condition for independence in algebraic matroid underlying higher determinantal varieties (Tsakiris 2020)
- Algebraic matroid of the FUNTF variety (B., Farnsworth, Rodriguez 2020)
- Algebraic matroids with graph symmetry (Király, Rosen, Theran 2013)
- Identifiability of latent tree mixtures and phylogenetic networks (Hollering and Sullivant 2020)
- Many other examples in rigidity theory

Matroids

Definition

A *matroid* is a pair $\mathcal{M} = (E, \mathcal{I})$ where E is a set and $\mathcal{I} \subseteq 2^E$ satisfies

1. $\mathcal{I}$ is nonempty,
2. if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
3. if $I, J \in \mathcal{I}$ with $|I| = |J| + 1$, then there exists $e \in I$ such that $J \cup \{e\} \in \mathcal{I}$.

Elements of $\mathcal{I}$ are called the *independent sets of* $\mathcal{M}$.

Definition

The *rank function* $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ maps $S \subseteq E$ to $|I|$ where I is the largest independent subset of S.

Definition

A *spanning set* of $\mathcal{M} = (E, \mathcal{I})$ is a set $S \subseteq E$ of maximum rank. A *basis* is a spanning independent set.

Example: graphic matroids

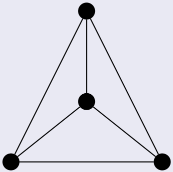
Definition

Let $G = (V, E)$ be a graph. In the *graphic matroid of* G,

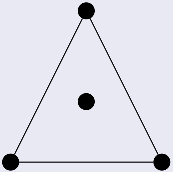
- $I \subseteq E$ is independent if (V, I) is a forest and a basis if a tree.
- The rank of $S \subseteq E$ is $|V| - c$ where c is the number of connected components of (V, S).

Note: when $G = K_n$, this is the algebraic matroid underlying CM_n^1.

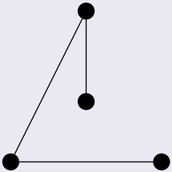
Example



G



Set of rank 2



Basis

Matroids from submodular functions

Definition (Edmonds and Rota 1966)

Let $f : 2^E \to \mathbb{Z}$ be increasing and submodular, i.e. satisfies

1. $f(A) \leq f(B)$ whenever $A \subseteq B \subseteq E$
2. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$.

Define $\mathcal{M}(f)$ to be the matroid on E where $I \subseteq E$ is independent iff

$$\text{for all } I' \subseteq I, \quad I' = \emptyset \quad \textit{or} \quad |I'| \leq f(I').$$

Example (Pym and Perfect 1970)

If $r_1, \ldots, r_d$ are rank functions of matroids $M_1, \ldots, M_d$ on ground set E, then I is independent in $\mathcal{M}(r_1 + \cdots + r_d)$ iff $I = I_1 \cup \cdots \cup I_d$ where I_j is independent in M_j.

Example (Lovász and Yemini 1982)

Let r be the rank function of the graphic matroid underlying K_n. Then $\mathcal{M}(2r - 1)$ is the algebraic matroid underlying CM_n^2.

Main theorem

Definition

Given $x, y \in \mathbb{C}^E$, the *Hadamard product* of x and y, denoted $x \star y$, has entries given by

$$(x \star y)_e := x_e y_e \quad \text{for all } e \in E.$$

Given varieties $U, V \subseteq \mathbb{C}^E$, the *Hadamard product $U \star V$ of U and V* is

$$U \star V := \overline{\{x \star y : x \in U, y \in V\}}.$$

Theorem (Main theorem)

Let $U, V \subseteq \mathbb{C}^E$ be linear spaces and let $r_U, r_V : 2^E \to \mathbb{Z}_{\geq 0}$ be the rank functions of their algebraic matroids. Then the algebraic matroid underlying $U \star V$ is $\mathcal{M}(r_U + r_V - 1)$.

Main theorem implies Lovász and Yemini's Theorem

Theorem (Lovász and Yemini)

Let r be the rank function of the algebraic matroid underlying CM_n^1. Then the algebraic matroid underlying CM_n^2 is $\mathcal{M}(2r-1)$.

proof sketch:

- A generic $\mathbf{d} \in CM_2^n$ can be expressed as $d_{uv} = (x_u - x_v)^2 + (y_u - y_v)^2$.
- Under the change of variables $z_u := x_u + iy_u$ and $w_u := x_u - iy_u$, this becomes $d_{uv} = (z_u - z_v)(w_u - w_v)$.
- Let $L \subseteq \mathbb{C}^{\binom{[n]}{2}}$ be the linear space parameterized as $d_{uv} = z_u - z_v$
- Note $CM_n^2 = L \star L$
- The algebraic matroid underlying L is the graphic matroid of K_n
- The result then follows from our main theorem

Laman's theorem

We will identify a graph G with its edge set. Number of vertices and connected components will be denoted $v(G)$ and $c(G)$.

Theorem (Pollaczek-Geiringer 1927)

A graph G is independent in the algebraic matroid underlying CM_n^2 if and only if each subgraph H of G satisfies $|H| \leq 2v(H) - 3$

proof sketch:

- Rank function r of graphic matroid is $r(G) = v(G) - c(G)$
- Lovász and Yemini's theorem implies that if G independent in CM_2^n and H is a subgraph, then $|H| \leq 2r(H) - 1 \leq 2v(H) - 3$
- If a subgraph H of G satisfies $|H| \geq 2v(H) - 2$, then some connected component H' of H satisfies $|H'| \geq 2v(H') - 2$
- Since H' connected, $2v(H') - 2 > 2r(H') - 1$

Tropical geometry

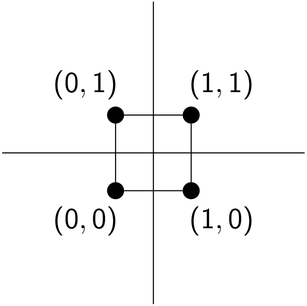
Definition

Given a complex polynomial f, the corresponding *tropical hypersurface*, $\mathrm{trop}(f)$, is the polyhedral fan consisting of the codimension-one cones in the normal fan of the Newton polytope of f.

The *tropicalization* of a variety $V \subseteq \mathbb{C}^E$, $\mathrm{trop}(V)$, is the intersection of all tropical hypersurfaces corresponding to polynomials f vanishing on V.

$$f = xy + x + y + 1$$

$$\mathrm{Newt}(f) = \mathrm{conv}\{(1,1),(1,0),(0,1),(0,0)\}$$



Tropical geometry

Theorem (Bieri, Groves, Bogart, Jensen, Speyer, Sturmfels, Thomas)

The tropicalization of an irreducible complex variety is the support of a pure balanced polyhedral fan that is connected through codimension 1.

Proposition (Yu 2016)

Tropicalization preserves the algebraic matroid structure of a variety.

More precisely, given any irreducible variety $V \subseteq \mathbb{C}^E$ *and* $S \subseteq E$,

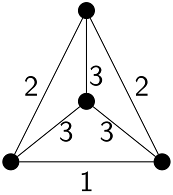
$$dim(\pi_S(V)) = \dim(\pi_S(\mathrm{trop}(V)))$$

Tropicalizing a linear space

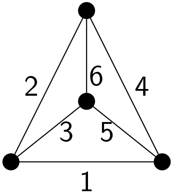
Definition

Let $\mathcal{M}$ be a matroid on ground set E. A weighting $w : E \to \mathbb{R}$ is called an *$\mathcal{M}$-ultrametric* if each $e \in E$ appears in a w-minimum basis. The set of all $\mathcal{M}$-ultrametrics, denoted $\tilde{\mathcal{B}}(\mathcal{M})$, is called the *Bergman fan of $\mathcal{M}$*.

Let $\mathcal{M}$ be the graphic matroid of K_4



An $\mathcal{M}$-ultrametric



Not an $\mathcal{M}$-ultrametric

Proposition

If $\mathcal{M}$ is the algebraic matroid of linear space $L \subseteq \mathbb{C}^E$ then $\operatorname{trop}(L) = \tilde{\mathcal{B}}(\mathcal{M})$.

Proof sketch of main theorem

Theorem

Let $U, V \subseteq \mathbb{C}^E$ be linear spaces and let $r_U, r_V : 2^E \to \mathbb{Z}_{\geq 0}$ be the rank functions of their algebraic matroids. Then the algebraic matroid underlying $U \star V$ is $\mathcal{M}(r_U + r_V - 1)$.

proof sketch:

- If $U, V \subseteq \mathbb{C}^E$ are varieties, then $\mathrm{trop}(U \star V) = \mathrm{trop}(U) + \mathrm{trop}(V)$.
- So if M and N are the algebraic matroids underlying U and V, $\mathrm{trop}(U \star V) = \tilde{\mathcal{B}}(M) + \tilde{\mathcal{B}}(N)$.
- Hard part: for $I \subseteq E$, show that there exists a cone of $\tilde{\mathcal{B}}(M) + \tilde{\mathcal{B}}(N) \subseteq \mathbb{C}^E$ whose coordinate projection onto $\mathbb{C}^I$ is $|I|$-dimensional iff I independent in $\mathcal{M}(r_U + r_V - 1)$.

Future directions

Question

Can we generalize the main theorem by replacing "linear spaces" with some larger family of irreducible varieties?

Cannot allow arbitrary varieties. If V can be parameterized by monomials, then $V \star V = V$.

Conjecture

Let $L_1, \ldots, L_d$ be linear spaces whose underlying matroids have rank functions $r_1, \ldots, r_d$. Then the algebraic matroid underlying $L_1 \star \cdots \star L_d$ is $\mathcal{M}(r_1 + \cdots + r_d - d + 1)$.

Question

What other varieties with interesting algebraic matroids can be expressed as Hadamard products?

The end

Thank you for your attention!

`https://arxiv.org/abs/2003.10529`